2

Relations, Schedules, and Objective Functions

When allocating scarce resources over time we have to define precedence relationships among the activities of the project. Those precedence relationships establish a binary relation in the activity set of the project. Together with the original temporal constraints, the binary relation gives rise to a preorder in the activity set. Depending on the type of basic project scheduling problem given and the specific objective function to be minimized, different types of preorders have to be investigated. In this chapter we review and extend a classification of schedules and objective functions that has been proposed by Neumann et al. (2000). The classification is based on two basic representations of the feasible region of project scheduling problems as unions of relation-induced polytopes. The purpose of the classification is to provide, for each class of objective functions, a finite set of candidates for optimal schedules that are characterized as specific points of the relation-induced polytopes such as minimal points, local minimizers of the objective function, or vertices.

2.1 Resource Constraints and Feasible Relations

Before we discuss the relationship between resource constraints and certain relations in the set of real activities or events, respectively, we first review some basic terminology.

Definition 2.1 (Binary relation, preorder, and strict order). *A binary relation ρ in (ground) set X is a set of pairs $(x, y) \in X \times X$. Relation ρ' in X with $\rho' \supseteq \rho$ is termed an extension of ρ. $tr(\rho)$ denotes the transitive hull of relation ρ, i.e., the $\subseteq$-minimal transitive extension of ρ in X. A transitive binary relation θ in set X is termed a preorder in X. Two elements $x, y \in X$ are referred to as comparable in preorder θ if $(x, y) \in \theta$ or $(y, x) \in \theta$, and incomparable, otherwise. θ is a complete preorder if $(i, j) \in \theta$ or $(j, i) \in \theta$ for all $i, j \in X$, $i \neq j$. A set $U \subseteq X$ of pairwise incomparable elements is called an antichain in θ. $Pred^{\theta}(x) = \{y \in X \mid (y, x) \in \theta\}$ is the set of predecessors of*

x in θ. $x \in Y \subseteq X$ is called a maximal element of Y in θ if $(y, x) \in \theta$ implies $(x, y) \in \theta$ for all $y \in Y$, $y \neq x$. An irreflexive preorder is asymmetric and thus represents a strict order. The covering relation $cr(\theta)$ of strict order θ is the $\subseteq$-minimal binary relation ρ in X with $tr(\rho) = \theta$. The precedence graph of strict order θ is the directed graph $G(\theta)$ with node set X and arc set $cr(\theta)$.

When we deal with renewable resources, forbidden sets F are broken up by introducing precedence constraints $S_j \geq S_i + p_i$ between real activities $i, j \in F$. In other words, we construct a strict order θ in the set V^a of real activities where $(i, j) \in \theta$ means that activity j cannot be started before activity i has been completed. In case of cumulative resources, surplus and shortage sets F are broken up by introducing precedence constraints $S_j \geq S_i$ between events $i \in V^e \setminus F$ and events $j \in F$. Thus, by resolving cumulative-resource conflicts we establish a reflexive preorder θ in event set V^e whose elements (i, j) say that event j cannot take place before the occurrence of event i.

The following two types of preorders will be needed when studying precedence relationships between real activities or events that are induced by a given schedule.

Definition 2.2 (Interval order and weak order). *An interval order in set X is a strict order θ in X for which $(w, x), (y, z) \in \theta$ implies $(w, z) \in \theta$ or $(y, x) \in \theta$ for all $w, x, y, z \in X$. A (reflexive) weak order in set X is a complete and reflexive preorder in X.*

2.1.1 Renewable-Resource Constraints

In this subsection we consider irreflexive relations in the set V^a of real activities for the scheduling of projects with renewable resources. We first define the concepts of time-feasible and feasible relations, which go back to the work of Radermacher (1978) and Bartusch et al. (1988). In difference to the treatment of the material by Neumann et al. (2000) and Neumann et al. (2003*b*), Sect. 2.3, we use relations instead of strict orders, which allows of a unifying view on renewable-resource and cumulative-resource constraints.

Definition 2.3 (Time-feasible and feasible relations). *Let ρ be an irreflexive relation in set V^a and let $\mathcal{S}_T(\rho) := \{S \in \mathcal{S}_T \mid S_j \geq S_i + p_i$ for all $(i, j) \in \rho\}$ be the set of all time-feasible schedules satisfying the precedence constraints given by ρ. $\mathcal{S}_T(\rho)$ is called the relation polytope of ρ. Relation ρ is termed time-feasible if $\mathcal{S}_T(\rho) \neq \emptyset$. A time-feasible relation ρ with $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$ is called feasible.*

Condition $\mathcal{S}_T(\rho) \neq \emptyset$ means that the precedence constraints from relation ρ do not contradict the prescribed temporal constraints. If $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$, all schedules satisfying those precedence constraints are feasible. If ρ is a feasible relation, then all time-feasible extensions $\rho' \supset \rho$ are feasible as well. A feasible relation ρ represents a solution to the *sequencing problem* of resource allocation, which consists in determining a (partial) order in which competing activities are processed on the resources. The subsequent *time-constrained project scheduling* of the activities is achieved by finding some (necessarily feasible) schedule $S \in \mathcal{S}_T(\rho)$ minimizing objective function f on $\mathcal{S}_T(\rho)$.

Let $\mathcal{M} \subseteq \mathcal{S}_T$ be a nonempty set of time-feasible schedules. We say that S is a *minimal point* of $\mathcal{M}$ if there is no $S' \in \mathcal{M}$ with $S' < S$, where $S' < S$ means $S' \leqq S$ and $S' \neq S$. Relation polytope $\mathcal{S}_T(\rho)$ is the set of all time-feasible schedules belonging to the following "expanded" project network $N(\rho)$. As a consequence, the corresponding earliest schedule represents the *unique* minimal point of polytope $\mathcal{S}_T(\rho)$ (see Subsection 1.1.3).

Definition 2.4 (Relation network). *Given relation ρ in set V^a, the relation network $N(\rho)$ results from project network N by adding, for each pair $(i,j) \in \rho$, the arc (i,j) with weight p_i. By $D(\rho) = (d^{\rho}_{ij})_{i,j \in V^a}$ we denote the distance matrix belonging to relation network $N(\rho)$.*

Bartusch et al. (1988) consider time-feasible strict orders θ that are extensions of the strict order

$$\Theta(D) := \{(i,j) \in V^a \times V^a \mid d_{ij} \geq p_i\}$$

in V^a induced by distance matrix D. We shall call such a strict order θ *BMR-feasible* if no antichain U in θ is forbidden. As we shall prove later on, the antichains in θ are exactly the sets of real activities which, subject to the precedence constraints from θ, can be in progress simultaneously. That is why any BMR-feasible strict order is feasible as well. On the other hand, there may be feasible strict orders $\theta \supseteq \Theta(D)$ which are not BMR-feasible, as will be illustrated in Example 2.10. The reason for this is that in general $\Theta(D(\theta)) \supset tr(\theta \cup \Theta(D))$. In the case where $\delta_{ij} \geq p_i$ for all $(i,j) \in E$, strict order θ is feasible precisely if $tr(\theta \cup \Theta(D)) = \Theta(D(\theta))$ is feasible.

By applying Theorem 1.17 we obtain the first basic representation of the set $\mathcal{S}$ of all feasible schedules.

Proposition 2.5 (Bartusch et al. 1988). *Let $\mathcal{MFR}$ be the set of all $\subseteq$-minimal feasible relations in activity set V^a. Then $\{\mathcal{S}_T(\rho) \mid \rho \in \mathcal{MFR}\}$ is a covering of $\mathcal{S}$.*

Notice that in general the above covering is not a partition of $\mathcal{S}$ because two different time-feasible relations ρ and ρ' may not be contradicting each other (i.e., $\mathcal{S}_T(\rho \cup \rho') = \mathcal{S}_T(\rho) \cap \mathcal{S}_T(\rho') \neq \emptyset$). Proposition 2.5 will be useful when dealing with objective functions that can efficiently be minimized on convex polytopes like regular or convex functions. In this case, the basic resource-constrained project scheduling problem (1.8) can be solved by enumerating (subsets of) relations $\rho \in \mathcal{MFR}$.

In the following we develop characterizations of time-feasible and feasible relations that allow for efficiently checking the feasibility of a given relation.

The latter technique will be used when dealing with the case of uncertain input data in Section 6.5, where solving a resource allocation problem requires the generation of appropriate feasible relations in the activity set. We shall apply a similar approach in Section 5.2 for deciding on the feasibility of schedules when resource units are occupied during a sequence-dependent changeover time between the execution of consecutive activities.

Proposition 2.6 (Neumann et al. 2000). *Relation ρ in V^a is time-feasible if and only if relation network $N(\rho)$ does not contain any directed cycle of positive length.*

Proof. By definition, relation ρ is time-feasible exactly if $\mathcal{S}_T(\rho) \neq \emptyset$. Polytope $\mathcal{S}_T(\rho)$ corresponds to the set of time-feasible schedules belonging to network $N(\rho)$. From Proposition 1.7 it follows that there is a time-feasible schedule for $N(\rho)$ precisely if $N(\rho)$ does not contain any directed cycle of positive length. □

As a consequence of Proposition 2.6, checking the time-feasibility of ρ can be done in $\mathcal{O}(n[m + |\rho|])$ time by applying Algorithm 1.1 to relation network $N(\rho)$ for computing distances d^{ρ}_{0i} for all $i \in V^a$. The next proposition shows how the feasibility of ρ can be established on the basis of distance matrix $D(\rho)$. We need the following preliminary lemma.

Lemma 2.7. *Let $\mathcal{S}_T \neq \emptyset$ and let $U \subseteq V^a$ be a set of real activities such that $d_{ij} < p_i$ for all $i, j \in U$. Then there exists a time-feasible schedule S with $\mathcal{A}(S, t) \supseteq U$ for some $t \geq 0$.*

Proof. Two activities $i, j \in U$ necessarily overlap in time if $d^{max}_{ij} < p_i$ and $d^{max}_{ji} < p_j$. Now assume that we add, for all $i, j \in U$ with $i \neq j$, a corresponding arc (j, i) weighted by $\delta_{ji} = -p_i + 1$ to project network N. We consider the addition of one of those arcs (j, i). $d_{ij} < p_i$ or, equivalently, $d_{ij} \leq p_i - 1$ implies $d_{ij} + \delta_{ji} \leq p_i - 1 - p_i + 1 = 0$. Proposition 1.9 then says that there is no directed cycle of positive length in the resulting (expanded) network. Moreover, for all modified distances d_{gh} with $g, h \in U$ we have $d_{gh} = d_{gj} + \delta_{ji} + d_{ih} = d_{gj} - p_i + 1 + d_{ih} \leq p_g - 1 - p_i + 1 + p_i - 1 = p_g - 1$ so that property $d_{gh} < p_g$ is preserved for all $g, h \in U$. Thus, after the addition of all arcs $(j, i) \in U \times U$ with $i \neq j$ there is no directed cycle of positive length in the resulting network N'. Proposition 1.7 then yields $\mathcal{S}'_T \neq \emptyset$ for the set $\mathcal{S}'_T$ of time-feasible schedules belonging to network N'. Due to the added maximum time lags, any two activities $i, j \in U$ overlap in time for each schedule $S \in \mathcal{S}'_T$, i.e., $[S_i, S_i + p_i[\cap [S_j, S_j + p_j[\neq \emptyset$ for all $i, j \in U$. The Helly property of intervals then implies that the interval $\cap_{i \in U}[S_i, S_i + p_i[$ during which all activities from set U overlap is nonempty for each $S \in \mathcal{S}'_T$. □

A constructive proof of Lemma 2.7 for the case where no deadline $\overline{d}$ for the latest termination of the project is prescribed can be found in Bartusch et al. (1988).

Proposition 2.8 (Neumann et al. 2003b, Sect. 2.3). *Time-feasible relation ρ in V^a is feasible if and only if for each minimal forbidden set $F \in \mathcal{F}$, relation network $N(\rho)$ contains a directed path of length $d^\rho_{ij} \geq p_i$ from some node $i \in F$ to some node $j \in F$.*

Proof. *Sufficiency*: Let ρ be a time-feasible relation such that for all minimal forbidden sets $F \in \mathcal{F}$, there is a pair (i, j) of activities $i, j \in F$ with $d^\rho_{ij} \geq p_i$. Each schedule $S \in \mathcal{S}_T(\rho)$ satisfies precedence constraint $S_j \geq S_i + p_i$ for all those pairs $(i, j) \in \Theta(D(\rho))$. From Theorem 1.17 it then follows that all schedules $S \in \mathcal{S}_T(\rho)$ are resource-feasible. Thus, with $\mathcal{S}_T(\rho) \subseteq \mathcal{S}_T$ we have $\emptyset \neq \mathcal{S}_T(\rho) \subseteq \mathcal{S}_R \cap \mathcal{S}_T = \mathcal{S}$.

Necessity: We assume that there is a forbidden set F with $d^\rho_{ij} < p_i$ for all $i, j \in F$. Then from Lemma 2.7 it follows that there exists a schedule $S \in \mathcal{S}_T(\rho)$ for which all activities $i \in F$ overlap in time. Thus, S is not resource-feasible and $\mathcal{S}_T(\rho) \nsubseteq \mathcal{S}$, which contradicts the feasibility of relation ρ. □

The following theorem is a direct consequence of Proposition 2.8.

Theorem 2.9. *Time-feasible relation ρ in V^a is feasible if and only if no antichain in strict order $\Theta(D(\rho))$ is forbidden.*

Proof. U is an antichain in $\Theta(D(\rho))$ exactly if $d^\rho_{ij} < p_i$ for all $i, j \in U$. Proposition 2.8 says that ρ is feasible if and only if no antichain in $\Theta(D(\rho))$ is a minimal forbidden set. Obviously, this is true exactly if no antichain is an (arbitrary) forbidden set because any forbidden antichain U would embed some minimal forbidden subchain $U' \subseteq U$. □

Theorem 2.9 implies that the feasibility of a time-feasible relation ρ can be verified by finding, for each $k \in \mathcal{R}^\rho$, a maximum-weight stable set U_k in the precedence graph $G(\theta)$ of strict order $\theta = \Theta(D(\rho))$ with weights r_{ik} for nodes $i \in V^a$. Since $G(\theta)$ is a transitive directed graph (see, e.g., Bang-Jensen and Gutin 2002, Sect. 1.8), such a set U_k can be determined efficiently by computing a minimum (s, t)-flow u^k in a flow network $\overline{G}_k(\theta)$ arising from $G(\theta)$ by adding two nodes s and t and arcs (s, i) and (j, t) for sources i and sinks j of $G(\theta)$ and where lower node capacities r_{ik} for nodes $i \in V^a$ have to be observed (cf. Kaerkes and Leipholz 1977 and Möhring 1985). This can be done in $\mathcal{O}(n^3)$ time by two applications of the FIFO preflow push algorithm for the maximum-flow problem with upper arc capacities (see, e.g., Ahuja et al. 1993, Sect. 7.7, or Bang-Jensen and Gutin 2002, Sect. 3.9). ρ is feasible precisely if for each $k \in \mathcal{R}^\rho$, the minimum-flow value $\phi(u^k)$ and thus the weight $\sum_{i \in U_k} r_{ik}$ of stable set U_k is less than or equal to resource capacity R_k.

Example 2.10. We consider a project with four real activities and one renewable resource. Figure 2.1a shows the relation network $N(\rho)$ belonging to strict order $\rho = \{(1, 2), (3, 4)\}$, where nodes $i \in V^a$ are labelled with durations p_i on the top and resource requirements r_i in boldface on the bottom. The resource capacity is $R = 2$. There are five minimal forbidden sets $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,4\}$, and $\{3,4\}$. ρ is *not* BMR-feasible because antichains $\{1,3\}$, $\{1,4\}$, and $\{2,4\}$ are forbidden sets. The strict order $\theta = \Theta(D(\rho))$ induced by distance matrix $D(\rho)$ equals $\{(1,2),(1,3),(1,4),(2,4),(3,4)\}$. The corresponding flow network $\overline{G}(\theta)$ is shown in Figure 2.1b. Each node i is labelled with lower node capacity r_i and each arc (i,j) is labelled with minimum flow u_{ij} on (i,j). A maximum-weight antichain in θ is $U = \{2,3\}$ whose weight $r_2 + r_3 = 2 \leq R$ equals the minimum flow value $\phi(u)$. Thus, strict order ρ is feasible.

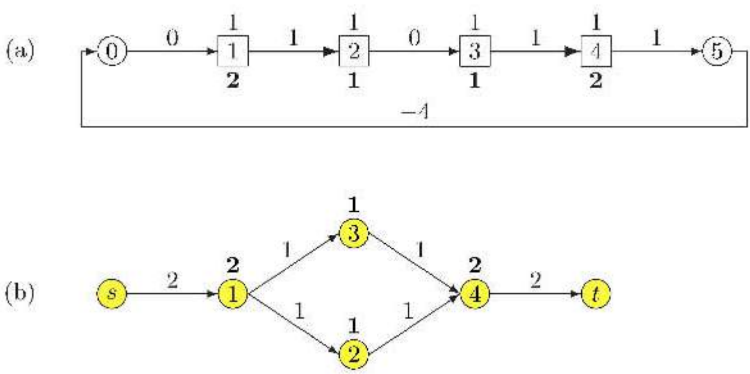


Fig. 2.1. Difference between feasibility and BMR-feasibility of strict orders: (**a**) relation network $N(\rho)$; (**b**) minimum (s,t)-flow in network $\overline{G}(\theta)$

We now turn to strict orders θ in V^a that are given by the precedence relationships induced by some schedule S.

Definition 2.11 (Schedule-induced strict order). *Given a schedule S, strict order $\theta(S) := \{(i,j) \in V^a \times V^a \mid S_j \geq S_i + p_i\}$ is the schedule-induced strict order which corresponds to the precedence relationships established by S. The relation polytope $\mathcal{S}_T(\theta(S))$ of $\theta(S)$ is called the schedule polytope of S, and the relation network $N(\theta(S))$ is called the schedule network of S.*

Schedule-induced strict orders $\theta(S)$ belong to the class of interval orders. This can be seen as follows. Let S be some schedule and let $(g,h),(i,j) \in \theta(S)$. If $(i,h) \notin \theta(S)$, then $S_j \geq S_i + p_i > S_h \geq S_g + p_g$, i.e., $(g,j) \in \theta(S)$.

By Definition 2.3 we have

$$\mathcal{S}_T(\rho) = \{S \in \mathcal{S}_T \mid \theta(S) \supseteq \rho\} \tag{2.1}$$

If schedule S is time-feasible, $\mathcal{S}_T(\theta(S))$ contains S. If schedule S is feasible, we have $\mathcal{S}_T(\theta(S)) \subseteq \mathcal{S}$. The reason for this is that all schedules $S' \in \mathcal{S}_T(\theta(S))$ satisfy $\theta(S') \supseteq \theta(S)$ (compare (2.1)) and thus each active set $\mathcal{A}(S',t')$ with

$0 \leq t' < \overline{d}$ is a subset of some active set $\mathcal{A}(S,t)$ where $0 \leq t < \overline{d}$. This proves the following proposition.

Proposition 2.12 (Neumann et al. 2000). *Strict order $\theta(S)$ induced by a time-feasible schedule S is feasible if and only if schedule S is feasible.*

Notice that for a time-feasible schedule S, strict order $\theta(S)$ represents *the* $\subseteq$-maximal relation whose relation polytope contains S. This can easily be shown by assuming the existence of some relation $\rho \supset \theta(S)$ with $S \in \mathcal{S}_T(\rho)$. Then relation ρ contains a pair $(i,j) \notin \theta(S)$. That is, we have $S_j < S_i + p_i$, which contradicts the assumption $S \in \mathcal{S}_T(\rho)$. The latter observation implies the following statement.

Proposition 2.13. *Each $\subseteq$-maximal feasible relation is induced by some feasible schedule.*

The relation polytope $\mathcal{S}_T(\theta)$ of some strict order θ is the set of all time-feasible schedules inducing an extension of θ. The set of all schedules inducing θ is termed the equal-order set of θ.

Definition 2.14 (Equal-order set). *Let θ be some schedule-induced strict order in set V^a. Equal-order set $\mathcal{S}_T^{=}(\theta) := \{S \in \mathcal{S}_T \mid \theta(S) = \theta\}$ is the set of all time-feasible schedules inducing strict order θ.*

Equal-order sets represent differences of schedule polytopes and thus are generally not closed. If θ is an $\subseteq$-maximal time-feasible strict order, we have $\mathcal{S}_T^{=}(\theta) = \mathcal{S}_T(\theta)$, and $\mathcal{S}_T^{=}(\theta) \subset \mathcal{S}_T(\theta)$, otherwise. Equal-order sets are convex because every schedule S on a line segment joining two schedules $S', S'' \in \mathcal{S}_T^{=}(\theta)$ induces strict order θ. The concept of equal-order sets leads to the second basic representation of the set $\mathcal{S}$ of all feasible schedules.

Proposition 2.15. *Let $\mathcal{SIO}$ be the set of all feasible schedule-induced strict orders. Then $\{\mathcal{S}_T^{=}(\theta) \mid \theta \in \mathcal{SIO}\}$ is a partition of $\mathcal{S}$.*

We will refer to this representation of $\mathcal{S}$ when dealing with resource levelling problems, where the objective function is regular or concave on equal-order sets and thus can be minimized by investigating minimal points or vertices, respectively, of equal-order sets. The following proposition shows that this corresponds to enumerating minimal points or vertices of schedule polytopes.

Proposition 2.16. *For a given project, the set of all minimal points (resp. vertices) of equal-order sets coincides with the set of all minimal points (resp. vertices) of schedule polytopes.*

Proof. We show the coincidence of the vertex sets. The same reasoning can be applied to minimal points. Let S be a vertex of some schedule polytope $\mathcal{S}_T(\theta)$. Then S is a vertex of equal-order set $\mathcal{S}_T^{=}(\theta(S))$ as well because $S \in \mathcal{S}_T^{=}(\theta(S))$ and $\mathcal{S}_T^{=}(\theta(S)) \subseteq \mathcal{S}_T(\theta)$. Now let S be a vertex of some equal-order set $\mathcal{S}_T^{=}(\theta)$. Then $\mathcal{S}_T^{=}(\theta) = \mathcal{S}_T(\theta(S)) \setminus (\cup_{\rho \supset \theta} \mathcal{S}_T(\rho))$. Since set $\cup_{\rho \supset \theta} \mathcal{S}_T(\rho)$ is closed, S must be a vertex of $\mathcal{S}_T(\theta(S))$. □

2.1.2 Cumulative-Resource Constraints

In this subsection we are concerned with relations establishing precedence relationships between the events of a project with cumulative resources. The concepts of time-feasible and feasible relations are defined in analogy to (time-) feasible relations for the case of renewable resources.

Definition 2.17 (Time-feasible and feasible relations). *Let ρ be a relation in event set V^e and let $\mathcal{S}_T(\rho) := \{S \in \mathcal{S}_T \mid S_j \geq S_i \text{ for all } (i,j) \in \rho\}$ be the relation polytope of ρ. Relation ρ is termed time-feasible if $\mathcal{S}_T(\rho) \neq \emptyset$. A time-feasible relation ρ is called feasible if $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$.*

A feasible relation in set V^e defines precedence constraints between the events from set V^e which are consistent with the temporal constraints and which ensure that all schedules $S \in \mathcal{S}_T(\rho)$ are feasible. The concepts of relation network $N(\rho)$ and corresponding distance matrix $D(\rho)$ are defined as for strict orders.

$$\Theta(D) := \{(i,j) \in V^e \times V^e \mid d_{ij} \geq 0\}$$

denotes the reflexive preorder in set V^e induced by distance matrix D.

Theorem 1.28 provides the first relation-based representation of the $\mathcal{S}$ of all feasible schedules.

Proposition 2.18. *Let $\mathcal{MFR}$ be the set of all $\subseteq$-minimal feasible relations in event set V^e. Then $\{\mathcal{S}_T(\rho) \mid \rho \in \mathcal{MFR}\}$ is a covering of $\mathcal{S}$.*

Again, the covering of $\mathcal{S}$ by relation polytopes is generally not a partition.

As for relations in set V^a, we investigate how the feasibility of a given relation in the event set can be checked efficiently. We need two preliminary lemmas. The first lemma shows that any event set $U \subseteq V^e$ arising from the union of predecessor sets in reflexive preorder $\Theta(D)$ can be an active set. The second lemma states that if not all minimal forbidden sets are broken up by precedence constraints induced by distance matrix D, then there exists a forbidden set satisfying the conditions of Lemma 2.19, which implies that there are time-feasible schedules which are not resource-feasible.

Lemma 2.19. *Let $\mathcal{S}_T \neq \emptyset$ and let $U \subseteq V^e$ be a set of events such that for all $i,j \in V^e$ with $d_{ij} \geq 0$, $j \in U$ implies $i \in U$. Then there exists a time-feasible schedule S with $\mathcal{A}(S,t) = U$ for some $t \geq 0$.*

Proof. We select some $j \in U$ with $d_{ji} \leq 0$ for all $i \in U$, e.g., a maximal element of U in reflexive preorder $\Theta(D)$. Since set U is finite, such a maximal element always exists. Event $i \in U$ necessarily occurs no later than j if $d_{ij}^{min} \geq 0$, and event $i \notin U$ must occur after j if $d_{ji}^{min} > 0$. Suppose that project network N is expanded by adding an arc (i,j) with weight $\delta_{ij} = 0$ for each $i \in U$, $i \neq j$ and by adding an arc (j,i) with weight $\delta_{ji} = 1$ for each $i \notin U$. In what follows we prove that the resulting network N' does not contain directed cycles of positive length. Event j has been chosen such that (1) $d_{jh} \leq 0$ for all $h \in U$.

Moreover, from the definition of set U it follows that (2) $d_{gh} \leq -1$ for all $g \notin U$, $h \in U$. We first consider the addition of one arc (i,j) with $i \in U$. Since (1) provides $d_{ji} \leq 0$, it follows from Proposition 1.9 that no directed cycle of positive length is created. Next we show that the updated distance matrix D still satisfies inequalities (1) and (2). Obviously, adding (i,j) does not change any distance d_{jh} with $h \in U$ since from (1) we have $d_{ji}+\delta_{ij}+d_{jh} \leq 0+0+d_{jh} = d_{jh}$. For distances d_{gh} with $g \notin U$ and $h \in U$ that are modified when calling Algorithm 1.3 we have $d_{gh} = d_{gi} + \delta_{ij} + d_{jh} \leq -1+0+0 = -1$ because of (1) and (2). Now consider the addition of one arc (j,i) where $i \notin U$. (2) provides $d_{ji} + d_{ij} \leq 1 + (-1) = 0$, and thus none of the created directed cycles has positive length. By applying (2) we obtain the inequality $d_{jh} = d_{jj} + \delta_{ji} + d_{ih} \leq 0 + 1 + (-1) = 0$ for the modified distances d_{jh} with $h \in U$. From (2) is also follows that $d_{gh} = d_{gj} + \delta_{ji} + d_{ih} \leq -1+1-1 = -1$ for the modified distances d_{gh} with $g \notin U$ and $h \in U$.

Thus, we can introduce a minimum time lag $d_{ij}^{min} = 0$ for all $i \in U$, $i \neq j$ and a minimum time lag $d_{ji}^{min} = 1$ for all $i \notin U$ such that the reduced set $\mathcal{S}'_T$ of time-feasible schedules belonging to expanded project network N' is nonempty. Since all events $i \in U$ occur before or at the same time as j and all events $i \notin U$ must be scheduled (strictly) later than j, the active set $\mathcal{A}(S, S_j)$ at time S_j coincides with set U for all schedules $S \in \mathcal{S}'_T$. □

Lemma 2.20. *If there is a minimal k-surplus set $F \in \mathcal{F}_k^+$ with $d_{ij} < 0$ for all $i \in V_k^{e^-} \setminus F$, $j \in F \cap V_k^{e^+}$ or a minimal k-shortage set $F \in \mathcal{F}_k^-$ with $d_{ij} < 0$ for all $i \in V_k^{e^+} \setminus F$, $j \in F \cap V_k^{e^-}$, then there exists a forbidden set F' for which $j \in F'$ implies $i \in F'$ for all $i, j \in V_k^{e^+} \cup V_k^{e^-}$ with $d_{ij} \geq 0$.*

Proof. Let F be a minimal k-surplus set with $d_{ij} < 0$ for all $i \in V_k^{e^-} \setminus F$, $j \in F \cap V_k^{e^+}$. We construct surplus set F' as follows. We first delete all $i \in V_k^{e^-} \cap F$ from F for which $d_{ij} < 0$ for all $j \in F \cap V_k^{e^+}$. Since for none of the deleted events i there is some $j \in F' \cap V_k^{e^+}$ with $d_{ij} \geq 0$, it holds that (1) $d_{ij} < 0$ for all $i \in V_k^{e^-} \setminus F'$, $j \in F' \cap V_k^{e^+}$. After the deletion of events i it holds that for any $h \in F'$ there is some $j \in F' \cap V_k^{e^+}$ with $d_{hj} \geq 0$. Now consider distances d_{ih} for $i \in V_k^{e^-} \setminus F'$ and $h \in V_k^{e^-} \cap F'$. For given $h \in V_k^{e^-} \cap F'$, let $j \in F' \cap V_k^{e^+}$ be an event such that $d_{hj} \geq 0$. (1) provides $0 < d_{ij} \leq d_{ih} + d_{hj}$ for all $i \in V_k^{e^-} \setminus F$, which together with $d_{hj} \geq 0$ implies $d_{ih} < 0$. Thus, we have (2) $d_{ih} < 0$ for all $i \in V_k^{e^-} \setminus F'$, $h \in F' \cap V_k^{e^-}$.

Next, we add all $j \in V_k^{e^+} \setminus F'$ to F' for which $d_{jj'} \geq 0$ for some $j' \in F' \cap V_k^{e^+}$, so that (3) $d_{gj} < 0$ for all $g \in V_k^{e^+} \setminus F'$, $j \in F' \cap V_k^{e^+}$. Let j be one of the added events and let $j' \in F' \cap V_k^{e^+}$ be an event such that $d_{jj'} \geq 0$. From (1) it follows that $0 > d_{ij'} \geq d_{ij} + d_{jj'}$ for all $i \in V_k^{e^-} \setminus F'$. Due to $d_{jj'} \geq 0$, this implies $d_{ij} < 0$ for all $i \in V_k^{e^-} \setminus F'$, and thus property (1) is preserved. The validity of property (2) is not affected by adding events $j \in V_k^{e^+} \setminus F'$ to F' either. Finally, consider distances d_{gh} for $g \in V_k^{e^+} \setminus F'$ and $h \in F' \cap V_k^{e^-}$. For given $h \in F' \cap V_k^{e^-}$, let $j \in F' \cap V_k^{e^+}$ be an event such that $d_{hj} \geq 0$. Using (1) we have $0 > d_{gj} \geq d_{gh} + d_{hj}$, which then implies $d_{gh} < 0$. Thus, it holds that (4) $d_{gh} < 0$ for all $g \in V_k^{e^+} \setminus F'$, $h \in F' \cap V_k^{e^-}$.

The resulting set F' is a surplus set because it arises from F by deleting events $i \in V_k^{e^-}$ and adding events $j \in V_k^{e^+}$. Moreover, from (1) to (4) we have $d_{ij} < 0$ for all $i \notin F'$ and all $j \in F'$, which proves the assertion. The case of a minimal k-shortage set F can be dealt with analogously. □

The next proposition, which translates the statement of Proposition 2.8 to the case of cumulative resources, characterizes the feasibility of relations on the basis of relation network $N(\rho)$.

Proposition 2.21. *Time-feasible relation ρ in V^e is feasible if and only if for each minimal k-surplus set $F \in \mathcal{F}_k^+$, relation network $N(\rho)$ contains a directed path of length $d_{ij}^\rho \geq 0$ from some node $i \in V_k^{e^-} \setminus F$ to some node $j \in F \cap V_k^{e^+}$ and for each minimal k-shortage set $F \in \mathcal{F}_k^-$, relation network $N(\rho)$ contains a directed path of length $d_{ij}^\rho \geq 0$ from some node $i \in V_k^{e^+} \setminus F$ to some node $j \in F \cap V_k^{e^-}$.*

Proof. Sufficiency: Let ρ be a time-feasible relation satisfying the conditions of Proposition 2.21. Since for each schedule $S \in \mathcal{S}_T(\rho)$ it holds that $S_j \geq S_i$ for all $(i, j) \in \Theta(D(\rho))$, Theorem 1.28 implies the resource-feasibility of all schedules $S \in \mathcal{S}_T(\rho)$. This means that $\mathcal{S}_T(\rho) \subseteq \mathcal{S}_C$ and thus $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$.

Necessity: We assume that for some resource $k \in \mathcal{R}^\gamma$, there is a k-surplus set F such that $d_{ij}^\rho < 0$ for all $i \in V_k^{e^-} \setminus F$, $j \in F \cap V_k^{e^+}$. Lemma 2.20 then provides some surplus set F' for which Lemma 2.19 establishes the existence of a time-feasible schedule S such that $\mathcal{A}(S,t) = F'$ for some $t \geq 0$, i.e., $\mathcal{S}_T(\rho) \not\subseteq \mathcal{S}$. □

Now we are ready to prove the counterpart of Theorem 2.9.

Theorem 2.22. *Time-feasible relation ρ in V^e is feasible if and only if no union of predecessor sets in $\Theta(D(\rho))$ is forbidden.*

Proof. Sufficiency: Let ρ be a time-feasible relation for which no union of predecessor sets in $\Theta(D(\rho))$ is forbidden. U is a union of predecessor sets in $\Theta(D(\rho))$ precisely if for all $i, j \in V^e$ with $d_{ij}^\rho \geq 0$, $j \in U$ implies $i \in U$. Since there does not exist any surplus set U with the latter property, Lemma 2.20 implies that for each minimal surplus set $F \in \mathcal{F}_k^+$, there are two events $i \in V_k^{e^-} \setminus F$ and $j \in F \cap V_k^{e^+}$ such that $d_{ij}^\rho \geq 0$. Symmetrically it holds that for each minimal shortage set $F \in \mathcal{F}_k^-$, there are two events $i \in V_k^{e^+} \setminus F$ and $j \in F \cap V_k^{e^-}$ with $d_{ij}^\rho \geq 0$. Proposition 2.21 then establishes the feasibility of ρ.

Necessity: For any union U of predecessor sets in $\Theta(D(\rho))$, it follows from Lemma 2.19 that there exists a schedule $S \in \mathcal{S}_T(\rho)$ with $\mathcal{A}(S,t) = U$ for some $t \geq 0$. If U is a forbidden set, schedule S is not resource-feasible, which means that $\mathcal{S}_T(\rho) \not\subseteq \mathcal{S}$. □

Next we discuss how the feasibility of a time-feasible relation ρ can be checked in polynomial time by using Theorem 2.22. The statement of the theorem can be reformulated in the following way: Time-feasible relation ρ in V^e is feasible precisely if for no $j \in V^e$ there is a forbidden union U of predecessor sets in $\theta = \Theta(D(\rho))$ containing j as maximal element of U in θ (compare proof of Lemma 2.19). For given $j \in V^e$, such a set U is defined by properties (1) $i \in U$ implies $h \in U$ for all $(h,i) \in \theta$, (2) $j \in U$, and (3) j is a maximal element of U in θ, i.e., for all $i \in U$, $(j,i) \in \theta$ implies $(i,j) \in \theta$. The latter condition is equivalent to $i \notin U$ for all $i \in V^e$ with $(j,i) \in \theta$ and $(i,j) \notin \theta$. Now let x_i be a binary decision variable indicating whether or not event $i \in V^e$ is contained in U. Then we have (1) $x_h \geq x_i$ for all $(h,i) \in \theta$, (2) $x_j = 1$, and (3) $x_i = 0$ for all $i \in V^e$ with $(j,i) \in \theta$ and $(i,j) \notin \theta$. The set U belonging to incidence vector $x = (x_i)_{i \in V^e}$ is forbidden exactly if for some $k \in \mathcal{R}^\gamma$, $\sum_{i \in U} r_{ik} < \underline{R}_k$ or $\sum_{i \in U} r_{ik} > \underline{R}_k$. Thus, the problem of testing the feasibility of ρ can be solved by verifying, for each event $j \in V^e$ and each resource $k \in \mathcal{R}^\gamma$, whether or not there exists a binary vector x satisfying constraints (1) to (3) such that $\sum_{i \in V^e} r_{ik} x_i$ is less than safety stock $\underline{R}_k$ or greater than storage capacity $\overline{R}_k$. For given event j and resource k, checking whether the storage capacity of k might be violated at the occurrence of j can be achieved by solving the following binary program.

$$
\left.
\begin{array}{lllr}
\text{Maximize} & \sum\limits_{i \in V^e} r_{ik} x_i & & \\
\text{subject to} & x_h - x_i \geq 0 & ((h,i) \in \theta : h \neq i) & (1) \\
& x_j = 1 & & (2) \\
& x_i = 0 & (i \in V^e : (j,i) \in \theta,\ (i,j) \notin \theta) & (3) \\
& x_i \in \{0,1\} & (i \in V^e) & (4)
\end{array}
\right\} \quad (2.2)
$$

The coefficient matrix of constraints (1) coincides with the negative transposed incidence matrix of the directed graph G_{jk} with node set V^e and arc set $\theta \setminus \{(i,i) \mid i \in V^e\}$. That is why the coefficient matrix of constraints (1) to (3) is totally unimodular, and the integrality condition (4) for variables x_i can be replaced with $0 \leq x_i \leq 1$ $(i \in V^e)$. As a consequence, problem (2.2) can be formulated as a linear program. In the sequel, we show that the dual of this linear program represents a minimum-flow problem.

Let $i \in V^e$ be some predecessor of j in θ. Then it follows from (1) and (2) that $x_i = 1$. Conversely, let j be predecessor of some $i \in V^e$ in θ with $(i,j) \notin \theta$. Then (3) implies that $x_i = 0$. The variables x_i with fixed value 1 or 0 can be eliminated as follows. If $x_i = 1$ because $(i,j) \in \theta$, the transitivity of reflexive preorder θ provides $(h,j) \in \theta$ and thus $x_h = 1$ for all $(h,i) \in \theta$. Symmetrically, assume that $x_h = 0$ because $(j,h) \in \theta$ and $(h,j) \notin \theta$. Then the transitivity of θ implies that $(j,i) \in \theta$ and $(i,j) \notin \theta$ and thus by (3) $x_i = 0$ for all $(h,i) \in \theta$. Hence, constraint (1) can be restricted to variables x_i for which $(i,j) \notin \theta$ and variables x_h for which $(j,h) \notin \theta$ or $(h,j) \in \theta$. Otherwise we would have $x_h = 1$ or $x_i = 0$, which implies (1). For those variables x_i

and x_h we can furthermore assume that $(j,i) \notin \theta$ and $(h,j) \notin \theta$ because else again $x_i = 0$ or $x_h = 1$. Now let $V_j^e := \{i \in V^e \mid (i,j),(j,i) \notin \theta\}$ be the set of all events i for which the value of x_i is not fixed in advance. Then constraint (1) needs only be considered for pairs $(h,i) \in \theta$ with $h \neq i$ and $h, i \in V_j^e$. V_j^e is the set of all $i \in V^e$ that are incomparable with j in θ. We note that due to Remark 1.6b, $\{0, n+1\} \cap V_j^e = \emptyset$ for all $j \in V_j^e$, and in particular $V_0^e = V_{n+1}^e = \emptyset$. By $\theta_j := \theta \cap (V_j^e \times V_j^e)$ we denote the sub-preorder of θ induced by set V_j^e. We obtain the following statement of problem (2.2) as a linear program, where the additive constant $\sum_{(i,j)\in\theta} r_{ik}$ is omitted in the objective function.

$$\left.\begin{array}{lll} \text{Maximize} & \sum\limits_{i \in V_j^e} r_{ik} x_i & \\ \text{subject to} & x_h - x_i \geq 0 & ((h,i) \in \theta_j : h \neq i) \\ & 0 \leq x_i \leq 1 & (i \in V_j^e) \end{array}\right\} \quad (2.3)$$

Now let s and t be a source and a sink to be added to directed graph G_{jk}. By $\overline{G}_{jk} = (\overline{V}_j, \overline{\theta}_j)$ where $\overline{V}_j := V_j^e \cup \{s,t\}$ and $\overline{\theta}_j := \theta_j \cup (\{s\} \times V_j^e) \cup (V_j^e \times \{t\})$ we denote the directed graph that results from G_{jk} by adding arcs (s,i) and (i,t) for all nodes $i \in V_j^e$. The dual of (2.3) can be formulated as the following minimum-flow problem in $\overline{G}_{jk}$ with supplies r_{ik} at nodes $i \in V_j^e$, where $\phi^j(u)$ denotes the value of flow u:

$$\left.\begin{array}{lll} \text{Minimize} & \phi^j(u) = \sum\limits_{i \in V_j^e} u_{it} & \\ \text{subject to} & \sum\limits_{(i,h)\in\overline{\theta}_j : h \neq i} u_{ih} - \sum\limits_{(h,i)\in\overline{\theta}_j : h \neq i} u_{hi} = r_{ik} & (i \in V_j^e) \\ & u_{hi} \geq 0 & ((h,i) \in \overline{\theta}_j : h \neq i) \end{array}\right\} \quad (2.4)$$

Problem (2.4) can be solved in $\mathcal{O}(n^3)$ time by first substituting supplies r_{ik} at nodes i into appropriate upper arc capacities (see, e.g., Bang-Jensen and Gutin 2002, Section 3.2) and then solving the minimum-flow problem with vanishing supplies (cf. Subsection 2.1.1). Let $\overline{u}^{jk}$ be some flow solving minimum-flow problem (2.4). Then the optimal objective function value for problem (2.2) equals $\sum_{(i,j)\in\theta} r_{ik} + \phi^j(\overline{u}^{jk})$, which is equal to the maximum inventory level in resource k at the occurrence of event j.

For testing whether the inventory might fall below the safety stock, we solve the minimum-flow problem where supplies r_{ik} at nodes $i \in V_j^e$ are replaced with $-r_{ik}$. With $\underline{u}^{jk}$ designating a corresponding minimum (s,t)-flow, the optimal objective function value for (2.2) with "Minimize" instead of "Maximize" equals $\sum_{(i,j)\in\theta} r_{ik} - \phi^j(\underline{u}^{jk})$, which coincides with the minimum inventory level in resource k at the occurrence of event j. ρ is feasible if for all events $j \in V^e$ and all resources $k \in \mathcal{R}^\gamma$,

$$\underline{R}_k + \phi^j(\underline{u}^{jk}) \leq \sum_{(i,j)\in\theta} r_{ik} \leq \overline{R}_k - \phi^j(\overline{u}^{jk})$$

In sum, checking the feasibility of a relation ρ takes $\mathcal{O}(|\mathcal{R}^\gamma|n^4)$ time (recall that the time-feasibility of ρ can be verified in $\mathcal{O}(n[m+|\rho|])$ time).

We illustrate the verification of feasibility for a relation by considering an example.

Example 2.23. Figure 2.2a shows a project network with five events and one cumulative resource for which we assume a safety stock of $\underline{R}=0$ and a storage capacity of $\overline{R}=2$. The node labels provide the respective resource requirements. We consider the empty relation $\rho=\emptyset$. The reflexive preorder induced by $D(\rho)=D$ is $\theta=\{(0,1),(0,2),(0,3),(0,4),(1,4),(2,3),(2,4),(3,2),(3,4)\} \cup \{(i,i) \mid i\in V^e\}$. When checking against the storage capacity for event $j=1$, we obtain the flow network $\overline{G}_1$ depicted in Figure 2.2b, where nodes $i\in V^e$ are labelled with supplies r_i. In the minimum (s,t)-flow $\overline{u}^1$, one unit is shipped from node 3 to node 2, and thus the minimum flow value $\phi^1(\overline{u}^1)$ equals 0 and $\sum_{(i,1)\in\theta} r_i+\phi^1(\overline{u}^1)=r_0+r_1+0=2$. Figure 2.2c shows the flow network $\overline{G}_2=\overline{G}_3$ belonging to events $j=2$ and $j=3$ with a minimum flow $\overline{u}^2=\overline{u}^3$ of value $\phi^2(\overline{u}^2)=\phi^3(\overline{u}^3)=2$ and $\sum_{(i,2)\in\theta} r_i+\phi^2(\overline{u}^2)=\sum_{(i,3)\in\theta} r_i+\phi^3(\overline{u}^3)=r_0+r_2+r_3+2=2$. By inverting the signs of the supplies, we obtain the minimum-flow problems for testing against the safety stock. The corresponding flow values are $\phi^1(\underline{u}^1)=0$ and $\phi^2(\underline{u}^2)=\phi^3(\underline{u}^3)=0$. Accordingly, we have $\sum_{(i,1)\in\theta} r_i-\phi^1(\underline{u}^1)=r_0+r_1-0=2$ and $\sum_{(i,2)\in\theta} r_i-\phi^2(\underline{u}^2)=\sum_{(i,3)\in\theta} r_i-\phi^3(\underline{u}^3)=r_0+r_2+r_3-0=0$, which shows the feasibility of relation ρ.

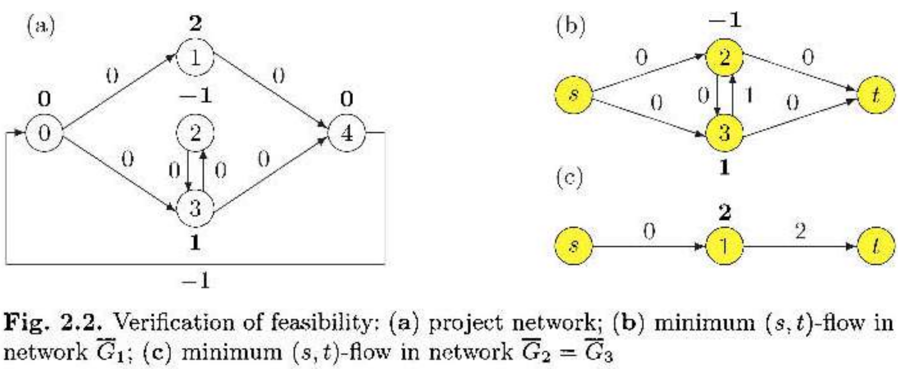


Fig. 2.2. Verification of feasibility: (**a**) project network; (**b**) minimum (s,t)-flow in network $\overline{G}_1$; (**c**) minimum (s,t)-flow in network $\overline{G}_2=\overline{G}_3$

We close this subsection by considering reflexive preorders in set V^e that are induced by some schedule. As we shall see, the results for schedule-induced strict orders in set V^a carry over to schedule-induced reflexive preorders in set V^e.

Definition 2.24 (Schedule-induced reflexive preorder). *Given a schedule S, reflexive preorder $\theta(S):=\{(i,j)\in V^e\times V^e \mid S_j\geq S_i\}$ is the schedule-induced reflexive preorder which corresponds to the precedence relationships* *established by S. $\mathcal{S}_T(\theta(S))$ is again called the schedule polytope of schedule S, and $N(\theta(S))$ is the schedule network of S.*

Due to their completeness, schedule-induced reflexive preorders $\theta(S)$ are reflexive weak orders. Proposition 2.12 saying that $\theta(S)$ is feasible precisely if S is feasible also applies to schedule-induced reflexive preorders. Analogously to Proposition 2.13 it can also be shown that the $\subseteq$-maximal feasible relations in set V^e are induced by feasible schedules. Let θ be some schedule-induced reflexive preorder in V^e and let equal-preorder set $\mathcal{S}_T^=(\theta) := \{S \in \mathcal{S}_T \mid \theta(S) = \theta\}$ again denote the set of all time-feasible schedules inducing θ. Similarly to the case of renewable resources, set $\mathcal{S}$ of all feasible schedules can again be represented as the union of nonintersecting equal-preorder sets.

Proposition 2.25. *Let $\mathcal{SIP}$ be the set of all feasible schedule-induced reflexive preorders. Then $\{\mathcal{S}_T^=(\theta) \mid \theta \in \mathcal{SIP}\}$ is a partition of $\mathcal{S}$.*

Again, it can be shown that each minimal point (resp. vertex) of an equal-preorder set is a minimal point (resp. vertex) of some schedule polytope and vice versa (see Proposition 2.16).

2.2 A Classification of Schedules

In machine and project scheduling without maximum time lags, different finite sets of minimal-point schedules have been used for the optimization of regular objective functions (see, e.g., Baker 1974, Sect. 7.2, for a study of nondelay, active, and semiactive schedules in machine scheduling and Sprecher et al. 1995 for the generalization of those concepts to project scheduling with renewable resources). Based on the feasible relations discussed in Subsection 2.1.1, the classification of Sprecher et al. (1995) has been extended by Neumann et al. (2000) to project scheduling problems with general temporal constraints and nonregular objective functions. This section refers to the latter classification of schedules.

All resource allocation methods discussed in this book are based on one of the two basic representations of set $\mathcal{S}$, either as a covering by relation polytopes or as partition by equal-preorder sets (where the term equal-preorder set may also designate an equal-order set). The schedules to be dealt with in Subsections 2.2.1 and 2.2.2 refer to the first and to the second representations, respectively.

2.2.1 Global and Local Extreme Points of the Feasible Region

Let $\mathcal{M} \subseteq \mathcal{S}_T$ be a nonempty set of time-feasible schedules. $S \in \mathcal{M}$ is a *(global) extreme point* of $\mathcal{M}$ if there are no two schedules $S', S'' \in \mathcal{M}$ such that $S = \alpha S' + (1 - \alpha)S''$ for some $0 < \alpha < 1$. If $\mathcal{M}$ is a polytope, each extreme point is a *vertex* of $\mathcal{M}$ and vice versa. We say that $S \in \mathcal{M}$ is a *local extreme point* of $\mathcal{M}$ if S is an extreme point of $\mathcal{M} \cap B_\varepsilon(S)$ for some $\varepsilon > 0$, where $B_\varepsilon(S) = \{S' \in \mathbb{R}^{n+2} \mid \|S - S\|_2 < \varepsilon\}$ is the ball of radius ε around S in $\mathbb{R}^{n+2}$. Recall that $S \in \mathcal{M}$ is a *minimal point* of $\mathcal{M}$ if there is no schedule $S' \in \mathcal{M}$ with $S' < S$. We notice that a minimal point of $\mathcal{M}$ need not represent a local extreme point of $\mathcal{M}$. As we will see later on, each minimal point of a relation polytope $\mathcal{M} = \mathcal{S}_T(\rho)$, however, is a local extreme point of $\mathcal{M}$.

Definition 2.26 (Active, stable, and pseudostable schedules). *A (feasible) schedule S is called active, stable, or pseudostable if S is a minimal point, an extreme point, or a local extreme point, respectively, of $\mathcal{S}$. $\mathcal{AS}$, $\mathcal{SS}$, and $\mathcal{PSS}$ denote the sets of all active, all stable, and all pseudostable schedules.*

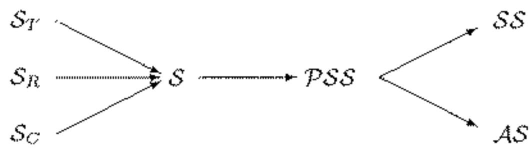
Active schedules have been introduced by Giffler and Thompson (1960) for solving open-shop problems with precedence constraints among operations and regular objective functions. In shop-floor scheduling, there is a one-to-one correspondence between job sequences on the machines and *semiactive schedules*, for which no operation can be processed earlier without changing the job sequences. Those semiactive schedules (as well as their analogues in project scheduling) are precisely the minimal points of components of $\mathcal{S}$, and every active schedule is semiactive.

Since each active, stable, or pseudostable schedule is a vertex of some relation polytope, the sets $\mathcal{AS}$, $\mathcal{SS}$, and $\mathcal{PSS}$ are finite. Neumann et al. (2000) provide an example of a project for which there is an active schedule that is not stable. However, each active schedule is pseudostable, which can be seen as follows. Assume that there exists some schedule $S \in \mathcal{AS} \setminus \mathcal{PSS}$. Since S is not pseudostable, we can find an open line segment ℓ passing through S that totally belongs to $\mathcal{S}$. The representation of $\mathcal{S}$ as a union of finitely many polytopes implies that ℓ can be chosen such that all points on ℓ are boundary points of $\mathcal{S}$, i.e., $\ell \subseteq \partial\mathcal{S}$. With $z \in [-1,1]^{n+2}$ being the direction of $\ell \subset S + \mathbb{R}z$, the minimality of S in $\mathcal{S}$ implies that $z \notin [0,1]^{n+2}$. It then follows from $\ell \subseteq \partial\mathcal{S}$ that all schedules on ℓ are minimal points, which contradicts the finiteness of $\mathcal{AS}$. Figure 2.3 summarizes the relationships between the schedule sets introduced.

In Neumann et al. (2000) it is shown by transformation from PARTITION that for the case of renewable resources, it is NP-hard to decide whether or not a given schedule is active, stable, or pseudostable. Since renewable-resource constraints can be expressed by temporal and cumulative-resource constraints without changing the order of magnitude of the problem size, this result also applies to project scheduling with cumulative resources.

2.2.2 Vertices of Relation Polytopes

All schedules considered in Subsection 2.2.1 represent vertices of $\subseteq$-*maximal* relation polytopes. We now turn to vertices of arbitrary relation polytopes.



Legend:

$\mathcal{A} \to \mathcal{B}$ means $\mathcal{A} \supseteq \mathcal{B}$

Fig. 2.3. Relationship between sets of schedules

Since each vertex of a relation polytope corresponds to some time-feasible schedule that is a vertex of its schedule polytope, we may restrict ourselves to (arbitrary) schedule polytopes.

Definition 2.27 (Quasiactive and quasistable schedules). *A feasible schedule S is called quasiactive or quasistable if S is the minimal point or a vertex, respectively, of its schedule polytope $\mathcal{S}_T(\theta(S))$. $\mathcal{QAS}$ and $\mathcal{QSS}$ denote the sets of all quasiactive and all quasistable schedules.*

Since the minimal point of a relation polytope is always a vertex, any quasiactive schedule is quasistable as well.

A schedule S is quasiactive precisely if no nonempty set of activities can be scheduled earlier without deleting at least one precedence relationship $(i, j) \in \theta(S)$ or violating some temporal constraint. Schedule S is quasistable exactly if there is no nonempty set of activities which can be scheduled both earlier and later such that all precedence relationships $(i, j) \in \theta(S)$ and all temporal constraints are observed. The next proposition provides an equivalent formulation of the latter observation, which will be useful when dealing with algorithms operating on the sets $\mathcal{QAS}$ and $\mathcal{QSS}$ of all quasiactive and all quasistable schedules in Chapter 4.

Proposition 2.28 (Neumann et al. 2000). *A feasible schedule S is*

(a) *quasiactive if and only if there exists a spanning outtree $G = (V, E_G)$ of its schedule network $N(\theta(S))$ rooted at node 0 such that $S_j - S_i = d_{ij}^{\theta(S)}$ for all arcs $(i, j) \in E_G$,*
(b) *quasistable if and only if there exists a spanning tree $G = (V, E_G)$ of its schedule network $N(\theta(S))$ such that $S_j - S_i = d_{ij}^{\theta(S)}$ for all arcs $(i, j) \in E_G$.*

From Proposition 2.28 it follows that the quasiactiveness and the quasistableness of a given schedule can be checked in polynomial time. A further implication of Proposition 2.28 is that any quasistable schedule (and thus any quasiactive schedule as well) is integral and that any quasiactive schedule S satisfies

$$S_{n+1} \leq \min\,(\overline{d},\ \sum_{i\in V} \max(\max_{(i,j)\in E} \delta_{ij}, p_i),\ \sum_{i\in V} \max(\max_{(h,i)\in E} \delta_{hi}, p_i))$$

Obviously, active schedules are quasiactive, and pseudostable schedules are quasistable. Figure 2.4 locates the quasiactive and quasistable schedules within the framework of the schedule sets introduced before.

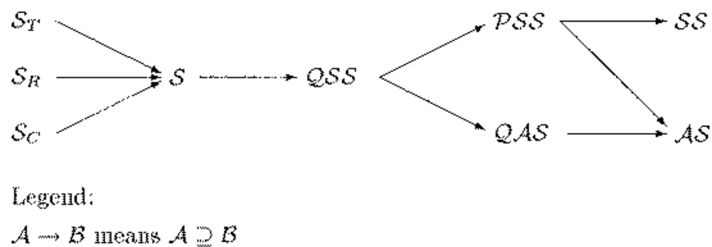


Fig. 2.4. Relationship between sets of schedules, revisited

2.3 Objective Functions

An objective function $f : \mathcal{S}_T \rightarrow \mathbb{R}$ associates each time-feasible schedule S with a numerical assessment $f(S)$. Recall that we have assumed f to be lower semicontinuous and thus f takes its minimum on compact set $\mathcal{S}$ if $\mathcal{S} \neq \emptyset$. Whereas *regular* objective functions f, which are componentwise nondecreasing, refer to temporal objectives of project planning like minimizing the project duration, *nonregular* objective functions typically translate some monetary goals such as minimizing inventory holding or capacity adjustment costs or maximizing the net present value of the project. In this section we are going to study several classes of objective functions, which cover a large variety of resource allocation problems in project management. Based on the results of Sections 2.1 and 2.2 we provide for each class a finite set of schedules containing at least one optimal schedule if $\mathcal{S} \neq \emptyset$. In Subsection 2.3.1 we consider objective functions that can be minimized efficiently on relation polytopes. Subsection 2.3.2 is concerned with objective functions for which in general already the time-constrained project scheduling problem is NP-hard. The latter objective functions are typically encountered when solving resource levelling problems, where the problem amounts to minimizing the variability in resource loading profiles of renewable resources (expressed in terms of range, variance, or total variation). Whereas resource allocation problems with objective functions from Subsection 2.3.1 can be solved by enumerating $\subseteq$-minimal feasible relations, minimizing objective functions from Subsection 2.3.2 requires the investigation of arbitrary schedule-induced preorders. For certain of the latter objective functions, however, the search for an optimal schedule can be limited to schedule polytopes belonging to $\subseteq$-maximal schedule-induced preorders. The latter objective functions will be studied in Subsection 2.3.3.

2.3.1 Regular and Convexifiable Objective Functions

Consider some nonempty relation polytope $\mathcal{S}_T(\rho)$. Any regular objective function is minimized by the unique minimal point $\min \mathcal{S}_T(\rho)$ of $\mathcal{S}_T(\rho)$, which coincides with the earliest schedule belonging to relation network $N(\rho)$. Now let f be some convex (and due to our lower semicontinuity assumption) continuous objective function. Then finding a minimizer of f on $\mathcal{S}_T(\rho)$ can, under some mild technical assumptions, be achieved in polynomial time, e.g., by the ellipsoid method (cf. Grötschel et al. 1998, Sect. 4.1) or, more efficiently on the average, by interior-point methods based on self-concordant barriers for $\mathcal{S}_T(\rho)$. Self-concordant barriers are available for different classes of convex functions (see the book by Nesterov and Nemirovskii 1994 for details). The next definition provides a class of objective functions which admits a smooth coordinate transformation such that the resulting time-constrained project scheduling problem is a convex programming problem. Recall that a bijection φ is called a C^1-diffeomorphism if both φ and φ^{-1} are continuously differentiable.

Definition 2.29 (Convexifiable and linearizable objective functions). *Let $f : \mathcal{S}_T \to \mathbb{R}$ be some objective function. We call f convexifiable if there exists a C^1-diffeomorphism $\varphi : \mathcal{S}_T \to X$ from $\mathcal{S}_T$ onto some Euclidean space X such that $f \circ \varphi^{-1}$ is a convex function and the images $\varphi(\mathcal{S}_T(\rho)) = \{\varphi(S) \mid S \in \mathcal{S}_T(\rho)\}$ of all relation polytopes under φ are convex sets. If $f \circ \varphi^{-1}$ is linear, we speak of a linearizable objective function f.*

Trivially, each convex objective function is convexifiable and each linear objective function is linearizable. In addition, we notice that due to the continuity of φ^{-1}, all images $\varphi(\mathcal{S}_T(\rho))$ are compact sets and because $\mathcal{S}_T$ is a relation polytope, set $X = \varphi(\mathcal{S}_T)$ is convex.

A time-feasible schedule $S \in \mathcal{M} \subseteq \mathcal{S}_T$ is called a *local minimizer* of f on $\mathcal{M}$ if for some $\varepsilon > 0$, S is a minimizer of f on the relative ball $\mathcal{M} \cap B_\varepsilon(S)$ around S in $\mathcal{M}$ (for the basic concepts of relative topology in Euclidean space needed for what follows we refer to Sydsæter et al. 1999, Ch. 12). Roughly speaking, the reason for the tractability of time-constrained project scheduling with convex objective functions is that each local minimizer of f on a relation polytope $\mathcal{S}_T(\rho)$ minimizes f on $\mathcal{S}_T(\rho)$. The next proposition relates the schedule sets introduced in Subsection 2.2.1 to regular and convexifiable objective functions. It also shows that, as for convex objective functions, any convexifiable objective function f can be minimized on relation polytopes $\mathcal{S}_T(\rho)$ by computing a local minimizer of f on $\mathcal{S}_T(\rho)$.

Proposition 2.30. *Let f be some lower semicontinuous objective function and assume that $\mathcal{S} \neq \emptyset$.*

(a) *If f is regular, the set of active schedules contains an optimal schedule.*
(b) *If f is linear, the set of stable schedules contains an optimal schedule.*
(c) *If f is linearizable, the set of pseudostable schedules contains an optimal schedule.*
(d) *If f is convexifiable, any set containing a local minimizer of f for each ($\subseteq$-maximal) relation polytope contains an optimal schedule.*

Proof. (a) and (b) are obvious. We first show (d). Let S be a local minimizer of f on some relation polytope $\mathcal{S}_T(\rho)$. Then there exists some $\varepsilon > 0$ such that $f(S) \leq f(S')$ for all $S' \in \mathcal{S}_T(\rho) \cap B_\varepsilon(S)$. With $x = \varphi(S)$ and $x' = \varphi(S')$ this means that $(f \circ \varphi^{-1})(x) = f(S) \leq f(S') = (f \circ \varphi^{-1})(x')$ for all $x' \in \varphi(\mathcal{S}_T(\rho) \cap B_\varepsilon(S))$. From the injectivity of φ we can infer that $\varphi(\mathcal{S}_T(\rho) \cap B_\varepsilon(S)) = \varphi(\mathcal{S}_T(\rho)) \cap \varphi(B_\varepsilon(S))$, where it follows from the continuity of φ^{-1} that $\varphi(B_\varepsilon(S))$ is open. As a consequence, there exists some $\varepsilon' > 0$ such that the ball $B'_{\varepsilon'}(x)$ with radius ε' around x in X is included in $\varphi(B_\varepsilon(S))$. This implies that x is a minimizer of $f \circ \varphi^{-1}$ on set $\varphi(\mathcal{S}_T(\rho)) \cap B'_{\varepsilon'}(x)$, i.e., a local minimizer of $f \circ \varphi^{-1}$ on $\varphi(\mathcal{S}_T(\rho))$. Since by assumption $f \circ \varphi^{-1}$ is a convex function and $\varphi(\mathcal{S}_T(\rho))$ is a convex set, x is also a (global) minimizer of $f \circ \varphi^{-1}$ on $\varphi(\mathcal{S}_T(\rho))$, i.e., $f(S) = (f \circ \varphi^{-1})(x) \leq (f \circ \varphi^{-1})(x')$ for all $x' \in \varphi(\mathcal{S}_T(\rho))$. Thus, we have $f(S) \leq f(S')$ for all S' with $x' = \varphi(S') \in \varphi(\mathcal{S}_T(\rho))$, or, equivalently, $f(S) \leq f(S')$ for all $S' \in \mathcal{S}_T(\rho)$. As a consequence, any local minimizer of f on some relation polytope $\mathcal{S}_T(\rho)$ minimizes f on the total polytope $\mathcal{S}_T(\rho)$. From Propositions 2.5 and 2.18 it follows that $\varphi(\mathcal{S}) = \varphi(\cup_{\rho \in \mathcal{MFR}} \mathcal{S}_T(\rho)) = \cup_{\rho \in \mathcal{MFR}} \varphi(\mathcal{S}_T(\rho)))$, which proves the assertion.

We now show statement (c). Since $f \circ \varphi^{-1}$ is linear on X, there exists some extreme point x of $\varphi(\mathcal{S}) \subseteq X$ that minimizes $f \circ \varphi^{-1}$ on $\varphi(\mathcal{S})$. Now assume that $S = \varphi^{-1}(x)$ is not a local extreme point of $\mathcal{S}$. Then there is an open line segment $\ell \subset \mathcal{S}$ containing S. Since φ^{-1} is continuous and φ is injective, this means that x is a relative interior point of $\varphi(\ell) \subset \varphi(\mathcal{S})$, which contradicts the fact that x is an extreme point of $\varphi(\mathcal{S})$. □

Neumann et al. (2000) have considered *quasiconcave* objective functions and so-called *binary-monotone* objective functions. An objective function f is said to be quasiconcave if its upper-level sets $U_\alpha = \{S \in \mathcal{S}_T \mid f(S) \geq \alpha\}$ are convex for every $\alpha \in \mathbb{R}$ (see, e.g., Avriel et al. 1988, Sect. 3.1). f is termed binary-monotone if f is nondecreasing or nonincreasing on each line segment in binary direction $z \in \{0,1\}^{n+2}$. A quasiconcave function attains its minimum on a compact set $\mathcal{M}$ at an extreme point of $\mathcal{M}$ because on closed line segments, the function is minimized at one of the two endpoints. That is why there always exists a stable schedule that minimizes f on set $\mathcal{S}$ if f is quasiconcave and $\mathcal{S} \neq \emptyset$. Since each relation polytope $\mathcal{S}_T(\rho)$ arises from the intersection of finitely many half spaces $\{S \in \mathbb{R}^{n+2}_{\geq 0} \mid S_0 = 0,\ S_j - S_i \geq d^\rho_{ij}\}$ where $(i,j) \in E \cup \rho$, binary-monotone objective functions, like the linearizable objective functions, always possess a vertex of $\mathcal{S}_T(\rho)$ among their minimizers on $\mathcal{S}_T(\rho)$. Thus, binary-monotone objective functions are minimized by pseudostable schedules. Unlike the case of convexifiable objective functions, however, a local minimizer of a quasiconcave or binary-monotone objective function f on some relation polytope is generally not a global minimizer of f on $S_T(\rho)$.

We proceed by providing examples of regular and convexifiable objective functions that are of interest in project scheduling. The simplest and most frequently used regular objective function is the *makespan* or *project duration*

$$f(S) = S_{n+1}$$

The project duration problem with renewable resources has been extensively studied in the literature during the four last decades (see Subsection 3.1.4 for an overview). Minimizing the project duration is a suitable objective if the majority of income payments occur at or after the end of the project, if the project deadline is tight and thus finishing the implementation of the project as early as possibly lowers the danger of exceeding the deadline, or if resource capacity is needed for future projects (cf. Kolisch 1995, Sect. 2.1).

A second regular objective function is the *total tardiness cost*

$$f(S) = \sum_{i \in V} w_i^t (S_i + p_i - d_i)^+$$

where $d_i \in \mathbb{Z}_{\geq 0}$ denotes a given due date for the completion of activity i and $w_i^t \in \mathbb{Z}_{\geq 0}$ is the cost arising from a late completion of activity i per unit time. This objective function is of particular interest for applications of resource allocation methods in make-to-order production scheduling, which will be discussed in Section 6.1. In that case, each real activity corresponds to the processing of a job on a machine, and violations of the delivery dates for the completed jobs incur conventional penalty per unit time.

We now turn to convexifiable objective functions. Of course, any linear and any convex objective function is convexifiable. A nonregular linear objective function is the *total inventory holding cost*

$$f(S) = \sum_{k \in \mathcal{R}^\gamma} c_k \int_0^{\overline{d}} r_k(S, t) dt$$

where we assume that each cumulative resource k stands for the inventory in a storage facility keeping one intermediate or final product with unit holding cost rate $c_k \in \mathbb{Z}_{\geq 0}$. Then $f(S)$ represents the cost arising from the stock in planning interval $[0, \overline{d}]$. The linearity of f can be seen as follows. A replenishment of resource k by r_{ik} units at time S_i incurs a holding cost of $c_k r_{ik}(\overline{d} - S_i)$. A depletion of k by $-r_{ik}$ units at time S_i saves a holding cost of $c_k(-r_{ik})(\overline{d} - S_i)$. Thus, the total inventory holding cost $f(S)$ can also be written as $\overline{d} \sum_{k \in \mathcal{R}^\gamma} c_k \sum_{i \in V^e} r_{ik} - \sum_{k \in \mathcal{R}^\gamma} c_k \sum_{i \in V^e} r_{ik} S_i$.

In general, certain activities and events i of a project are associated with a cash flow $c_i^f \in \mathbb{Z}$, which may be a paying out for raw materials or workforce or a paying in arising at the completion of a task when reaching a milestone.

When evaluating the profitability of a long-term project, the cash flows have to be discounted by some interest rate α, which can, e.g., be chosen to be the minimum attractive rate of return. The sum of all cash flows discounted to time 0 is called the *net present value* of the project. For the sake of simplicity, we suppose that all cash flows are discounted continuously and that each cash flow c_i^f arises at time S_i. The factor by which cash flow c_i^f is discounted then equals $e^{-\alpha S_i}$, and thus the net present value depends on the schedule S according to which the project is performed. By minimizing the negative net present value

$$f(S) = -\sum_{i\in V} c_i^f\, e^{-\alpha S_i}$$

we obtain a schedule that maximizes the financial benefit of the project in terms of its net present value. Grinold (1972) has shown that the (negative) net present value is a linearizable objective function. Let $\varphi : \mathcal{S}_T \to X \subseteq \mathbb{R}^{n+2}$ be defined as $\varphi(S) = (\varphi_i(S))_{i\in V}$ where $\varphi_i(S) = e^{-\alpha S_i}$. With $x_i = \varphi_i(S)$, the temporal constraints $S_j - S_i \geq \delta_{ij}$ can be stated as $x_j - e^{-\alpha\delta_{ij}} x_i \leq 0$ and $S_0 = 0$ becomes $x_0 = 1$. The linearized objective function is $(f \circ \varphi^{-1})(x) = -\sum_{i\in V} c_i^f x_i$. In addition, the net present value function f is binary-monotone because f is differentiable and for any time-feasible schedule S and any binary direction $z \in \{0,1\}^{n+2}$, the directional derivative of f at a point $S + \sigma z \in \mathcal{S}_T$ in direction z is $df|_{S+\sigma z}(z) = e^{-\alpha\sigma} df|_S(z)$ (see Subsection 3.2.2 and Neumann et al. 2003*b*, Sect. 3.3).

A convex objective function considered in project management is the *total earliness-tardiness cost*

$$f(S) = \sum_{i\in V} (w_i^e [d_i - S_i - p_i]^+ + w_i^t [S_i + p_i - d_i]^+)$$

where w_i^e and w_i^t respectively denote the cost per unit time incurred by an early or a late completion of activity $i \in V$ with respect to given due date $d_i \in \mathbb{Z}_{\geq 0}$ (see, e.g., Schwindt 2000*c* or Vanhoucke et al. 2001). Another example of a convex objective function is the negative *total weighted free float* of the project

$$f(S) = \sum_{i\in V} w_i^f \big(\max_{(j,i)\in E} [S_j + \delta_{ji}] - \min_{(i,j)\in E} [S_j - \delta_{ij}]\big)$$

For given schedule S, the total weighted free float of the project is the weighted sum of all early and late free floats of activities $i \in V$ if the earliest and latest start times ES_i and LS_i are set to be equal to S_i (cf. Subsection 1.1.3). A schedule with maximum total weighted free float can be regarded as robust in the sense that when executing the project, deviations of individual start times S_i from schedule will minimally affect the start times of other activities. In Section 6.5 we shall discuss how the total weighted earliness-tardiness and total weighted free float objective functions can be used for project scheduling under uncertainty.

Before concluding this subsection, we notice that all objective functions discussed above are continuous, which of course implies their lower semicontinuity.

2.3.2 Locally Regular and Locally Concave Objective Functions

In this subsection we move on to objective functions that are regular or concave on individual equal-preorder sets. Those objective functions play an important role for resource levelling, where one strives at smoothing loading profiles $r_k(S,\cdot)$ of renewable resources $k \in \mathcal{R}^\rho$ over time. Resource levelling problems typically arise when resource capacities may, at a certain cost, be adapted to the respective requirements. In that case, the resource capacities are regarded as being unlimited and the problem is to find a feasible minimum-cost schedule. However, besides the cost point of view, levelling loading profiles over time is of interest in its own right because in practice, evenly used resources tend less to be subject to disruption than resources whose usage is highly fluctuating over time. Accordingly, it has been proposed to use resource levelling as a technique for capacitated master production scheduling in production planning, where for a planning horizon of about one year, the monthly production quantities matching the gross requirements for the main products of a company are determined (see Franck et al. 1997, Neumann and Schwindt 1998, and Section 6.2).

Definition 2.31 (Locally regular and locally concave objective functions). *Let $f : \mathcal{S}_T \to \mathbb{R}$ be some objective function. We call f locally regular, if f is regular on all equal-preorder sets. f is termed locally concave if f is concave on all equal-preorder sets.*

The following proposition establishes the connection between locally regular and locally concave objective functions and the sets of quasiactive and quasistable schedules introduced in Subsection 2.2.2.

Proposition 2.32 (Neumann et al. 2000). *Let f be some lower semicontinuous objective function and assume that $\mathcal{S} \neq \emptyset$.*

(a) *If f is locally regular, the set of quasiactive schedules contains an optimal schedule.*
(b) *If f is locally concave, the set of quasistable schedules contains an optimal schedule.*

Proof. The lower semicontinuity of f and the compactness of $\mathcal{S}$ imply that f attains its minimum on $\mathcal{S}$. We first show statement (a). From the regularity of f on equal-preorder sets we can conclude that this minimum is taken at the minimal point of some equal-preorder set, which at the same time represents the minimal point of some schedule polytope (see Proposition 2.16, which applies to cumulative resources as well). We now show statement (b). From the concavity of f on equal-preorder sets it follows that f assumes its minimum at a vertex of some equal-preorder set. Proposition 2.16 says that this vertex is also a vertex of a schedule polytope. □

In contrast to regular or convexifiable objective functions, locally regular and locally concave objective functions cannot be minimized efficiently on relation polytopes in general. In particular this means that a resource allocation problem with a locally regular or a locally concave objective function generally does not become more tractable when the resource constraints are deleted. Below we shall give an example of a locally regular objective function for which time-constrained project scheduling is NP-hard. Note that minimizing such a function on an equal-preorder set constitutes an easy (though possibly unsolvable) problem because any equal-preorder set possesses at most one minimal point. Concerning locally concave functions, it is well-known that already the minimization of concave functions on hypercubes is NP-hard (cf. Horst and Tuy 1996, Sect. A.1.2). Proposition 2.28 indicates a simple way of generating all quasiactive or all quasistable schedules by constructing all spanning outtrees rooted at node 0 (resp. spanning trees) of relation networks belonging to feasible schedule-induced preorders. A corresponding schedule-generation scheme will be discussed in Section 4.1.

Next we consider locally regular and locally concave objective functions of resource levelling problems that have been discussed in literature. The objective functions express the variability in the utilization of renewable resources over time in terms of the range, the variance, and the total variation, respectively, of the loading profiles $r_k(S,\cdot)$ of renewable resources $k \in \mathcal{R}^\rho$.

An example of a locally regular objective function is the *total procurement cost* for renewable resources

$$f(S) = \sum_{k \in \mathcal{R}^\rho} c_k \max_{0 \le t \le \bar{d}} r_k(S,t)$$

where $c_k \in \mathbb{Z}_{\ge 0}$ denotes the unit procurement cost of renewable resource $k \in \mathcal{R}^\rho$. The total procurement cost equals the weighted sum of the maximum resource requirements (or, in other words, the weighted sum of the *ranges* of the loading profiles $r_k(S,\cdot)$).

Proposition 2.33. *The total procurement cost f is a lower semicontinuous and locally regular objective function.*

Proof. The lower semicontinuity can be seen as follows. Let S be some time-feasible schedule. The closedness of relation polytopes $\mathcal{S}_T(\rho)$ with $\rho \not\subseteq \theta(S)$ implies that there exists some $\varepsilon > 0$ such that $\theta(S') \subseteq \theta(S)$ for all S' contained in the relative ball $B_\varepsilon(S) \cap \mathcal{S}_T$ in $\mathcal{S}_T$ around S. Since for each resource $k \in \mathcal{R}^\rho$, $\max_{0 \le t \le \bar{d}} r_k(S,t)$ coincides with the weight of a maximum-weight antichain in $\theta(S)$, we obtain $f(S') \ge f(S)$ for all $S' \in B_\varepsilon(S) \cap \mathcal{S}_T$. The lower semicontinuity now follows from the fact that f is lower semicontinuous precisely if $f(S) \le \liminf_{S' \to S} f(S')$ for all $S \in \mathcal{S}_T$ (see, e.g., Hiriart-Urruty and Lemaréchal

1993, Sect. A.1). Since $f(S)$ equals the weight of a maximum-weight antichain in $\theta(S)$, f is constant and thus regular on equal-order sets. □

The total procurement cost is the objective function of the resource investment problem introduced by Möhring (1984). The resource investment problem arises in applications where installing resources incurs fixed transportation or setup costs per unit capacity. The recognition version (i.e., the question whether there is a feasible solution whose objective function value is smaller than or equal to a given threshold value, see, e.g., Papadimitriou and Steiglitz 1998, Sect. 15.2) of a resource investment problem with one resource coincides with the feasibility version (i.e., the question whether there is a feasible solution) of the corresponding resource-constrained project duration problem. The latter decision problem has been shown to be NP-complete by Theorem 1.12, which implies that the resource investment problem is NP-hard even if $R_k = \infty$ for all $k \in \mathcal{R}^\rho$. A classical objective function in the field of resource levelling that has been studied since the early work of Burgess and Killebrew (1962) is the *total squared utilization cost* for renewable resources

$$f(S) = \sum_{k \in \mathcal{R}^\rho} c_k \int_0^{\overline{d}} r_k^2(S,t)dt$$

where $c_k \in \mathbb{Z}_{\geq 0}$. Since workload $\int_0^{\overline{d}} r_k(S,t)dt = \sum_{i \in V^a} r_{ik} p_i$ does not depend on schedule S, $f(S)$ equals the weighted sum of the *variances* of the loading profiles $r_k(S,\cdot)$ plus a constant.

Proposition 2.34. *The total squared utilization cost f is a lower semicontinuous and locally concave objective function.*

Proof. The lower semicontinuity of f follows from its continuity. We show that f is concave on equal-order sets. For given schedule S, let $\mathcal{AC}(S)$ be the set of antichains in strict order $\theta(S)$, let $r_k(U) = \sum_{i \in U} r_{ik}$ be the weight of antichain $U \in \mathcal{AC}(S)$, and let $p(U,S) = \int_{t:\mathcal{A}(S,t)=U} dt$ be the time during which precisely the activities $i \in U$ overlap in time given schedule S. By $w_k(U,S) = r_k(U)p(U,S)$ we denote the corresponding workload on resource $k \in \mathcal{R}^\rho$. The total squared utilization cost can then be written as $f(S) = \sum_{k \in \mathcal{R}^\rho} c_k \sum_{U \in \mathcal{AC}(S)} r_k(U) w_k(U,S)$.

Now consider two schedules S and S' inducing the same strict order $\theta(S) = \theta(S')$. For any $\alpha \in [0,1]$ we have $\mathcal{AC}(S) = \mathcal{AC}(S') = \mathcal{AC}(\alpha S + (1-\alpha)S')$. With respect to schedule S, the activities i from a nonempty antichain $U \in \mathcal{AC}(S)$ overlap during $\overline{p}(U,S) = \min_{i \in U} C_i - \max_{i \in U} S_i > 0$ units of time, where $\overline{p}(U,S) = p(U,S)$ if U is $\subseteq$-maximal in $\mathcal{AC}(S)$. Since function $\overline{p}(U,\cdot)$ is concave on $S_T^=(\theta(S))$, we have $\overline{p}(U, \alpha S + (1-\alpha)S') \geq \alpha \overline{p}(U,S) + (1-\alpha)\overline{p}(U,S')$. Consequently, $w_k(U, \alpha S + (1-\alpha)S') \geq \alpha w_k(U,S) + (1-\alpha) w_k(U,S')$ for all $\subseteq$-maximal antichains $U \in \mathcal{AC}(S)$ and all $k \in \mathcal{R}^\rho$. As $\sum_{U \in \mathcal{AC}(S)} w_k(U,S) = \sum_{U \in \mathcal{AC}(S)} w_k(U,S') = \sum_{U \in \mathcal{AC}(S)} w_k(U, \alpha S + (1-\alpha)S') = \sum_{i \in U} r_{ik} p_i$ for all $k \in \mathcal{R}^\rho$, a positive difference $w_k(U, \alpha S + (1-\alpha)S') - [\alpha w_k(U, S) + (1-\alpha) w_k(U, S')]$ for the latter antichains U weighted by $r_k(U)$ corresponds to an equally large negative difference for the remaining (not $\subseteq$-maximal) antichains $U' \subset U$ weighted by $r_k(U') \leq r_k(U)$. By recursively applying the above reasoning to the function which arises from f by deleting the $\subseteq$-maximal elements from set $\mathcal{AC}(S)$ until $\mathcal{AC}(S) = \emptyset$, we finally obtain $f(\alpha S + (1-\alpha)S') \geq \alpha f(S) + (1-\alpha) f(S')$ for any $\alpha \in [0, 1]$, which provides the concavity of f on equal-order sets. □

By transformation from 3-Partition, Neumann et al. (2003*b*), Sect. 3.4, have shown that finding a time-feasible schedule with minimum total squared utilization cost is NP-hard.

Now let $t_1 < \cdots < t_\nu$ denote the start and completion times of real activities $i \in V^a$. Any jump discontinuity in loading profiles $r_k(S, \cdot)$ for $k \in \mathcal{R}^\rho$ occurs at some start or completion time t_μ where $1 \leq \mu \leq \nu$. A further resource-levelling objective function that has been studied in literature is the *total adjustment cost* for renewable resources

$$f(S) = \sum_{k \in \mathcal{R}^\rho} c_k \sum_{\mu=1}^{\nu} |r_k(S, t_\mu) - r_k(S, t_{\mu-1})|$$

where $t_0 := -1$ and $c_k \in \mathbb{Z}_{\geq 0}$ is the cost arising from increasing or decreasing the availability of resource $k \in \mathcal{R}^\rho$ by one unit (see, e.g., Younis and Saad 1996 or Neumann and Zimmermann 2000). Note that since $r_k(S, t_0) = r_k(S, t_\nu) = 0$ for all $k \in \mathcal{R}^\rho$, $f(S)$ equals $2 \sum_{k \in \mathcal{R}^\rho} c_k \sum_{\mu=1}^{\nu} [r_k(S, t_\mu) - r_k(S, t_{\mu-1})]^+$. Thus, the case where decreasing the availability of (certain) resources does not incur additional cost is contained in the total adjustment cost problem. The total adjustment cost coincides with the weighted sum of the *total variations* of the loading profiles $r_k(S, \cdot)$.

Proposition 2.35. *The total adjustment cost f is a lower semicontinuous and locally concave objective function.*

Proof. Apparently, $f(S)$ can be expressed as a function of all pairs $(i, j) \in \theta(S)$ for which the precedence constraints $S_j \geq S_i + p_i$ are active. Consequently, f is constant on the relative interior of any face of an equal-order set. Moreover, it is easily seen that for any such face, the objective function values of relative boundary points are less than or equal to the objective function values of corresponding relative interior points. Hence, f is concave on equal-order sets and lower semicontinuous. □

Finally, we notice that in contrast to the total procurement and total squared utilization costs, f is in general not continuous on equal-order sets. In Neumann et al. (2003*b*), Sect. 3.4, it is shown by the same polynomial transformation from 3-Partition as for the total squared utilization cost that minimizing the total adjustment cost on set S_T is NP-hard as well.

2.3.3 Preorder-Decreasing Objective Functions

In certain cases, the number of schedules to be enumerated for minimizing a locally regular or a locally concave objective function can be decreased by restricting the search to schedules inducing a maximum number of precedence relationships.

Definition 2.36 (Preorder-decreasing objective function). *An objective function f is called preorder-decreasing if $\theta' \supseteq \theta$ implies* $\inf_{S \in \mathcal{S}_T^{=}(\theta')} f(S) \leq \inf_{S \in \mathcal{S}_T^{=}(\theta)} f(S)$ *for all schedule-induced preorders θ and θ'.*

It follows from the definition of preorder-decreasing objective functions that, if $\mathcal{S} \neq \emptyset$, such functions possess a minimizer on some schedule polytope belonging to an $\subseteq$-maximal schedule-induced preorder. The total procurement cost is an example of a preorder-decreasing objective function, as has already been noticed by Möhring (1984). As an alternative to the construction of spanning trees (see preceding Subsection 2.3.2), a preorder-decreasing locally regular or locally concave objective function can be minimized on $\mathcal{S}$ by generating the set of $\subseteq$-maximal feasible schedule-induced preorders. Nübel (1999) has proposed a branch-and-bound algorithm for the resource investment problem that is implicitly based on this concept. The approach generally proves advantageous if the minimization of the objective function on equal-preorder sets already constitutes an NP-hard problem (which in particular may be the case for locally concave objective functions) because only the vertices of the generated $\subseteq$-minimal schedule polytopes have to be investigated.

In conclusion, Table 2.1 summarizes the relationships between the different classes of objective functions introduced and the sets of candidate schedules discussed in Section 2.2.

Table 2.1. Objective functions f and minimizers on $\mathcal{S}$

Objective function	Minimizer
Regular	Minimal point of $\mathcal{S}$
Convexifiable	Local minimizer on $\subseteq$-max. relation polytope $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$
Locally regular	Minimal point of schedule polytope $\mathcal{S}_T(\theta(S)) \subseteq \mathcal{S}$
Locally concave	Vertex of schedule polytopes $\mathcal{S}_T(\theta(S)) \subseteq \mathcal{S}$
Preorder-decreasing locally regular	Minimal point of $\subseteq$-minimal schedule polytopes $\emptyset \neq \mathcal{S}_T(\theta(S)) \subseteq \mathcal{S}$
Preorder-decreasing locally concave	Vertex of $\subseteq$-minimal schedule polytopes $\emptyset \neq \mathcal{S}_T(\theta(S)) \subseteq \mathcal{S}$